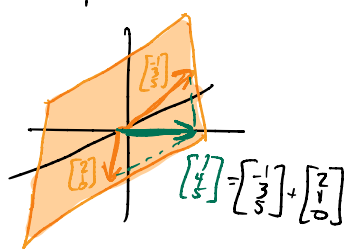
Column Space

Let A be an $m \times n$ matrix. The **Column Space of A**, $C(A)$, is the span of the columns of A

Ex: $A = \begin{bmatrix} 2 & -1 \\ 1 & 3 \\ 0 & 5 \end{bmatrix}$.

$$C(A) = \left\{ x_1 \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} -1 \\ 3 \\ 5 \end{bmatrix} \;\middle|\; x_1, x_2 \in \mathbb{R} \right\} = \left\{ \begin{bmatrix} 2x_1 - x_2 \\ x_1 + 3x_2 \\ 5x_2 \end{bmatrix} \;\middle|\; x_1, x_2 \in \mathbb{R} \right\} \subset \mathbb{R}^3$$

Geometrically, this is a plane in $\mathbb{R}^3$ since the columns are not multiples of each other.



Key Feature: If A is $m \times n$, $C(A)$ is a subspace of $\mathbb{R}^m$ (since it is the span of vectors in $\mathbb{R}^m$)

Why study $C(A)$? It is related to systems.

Let $A = [a_1 \cdots a_n]$ (i^{th} column of A is a_i). Then

$$C(A) = \{ x_1 a_1 + \cdots + x_n a_n \mid x_1, \ldots, x_n \in \mathbb{R} \}$$

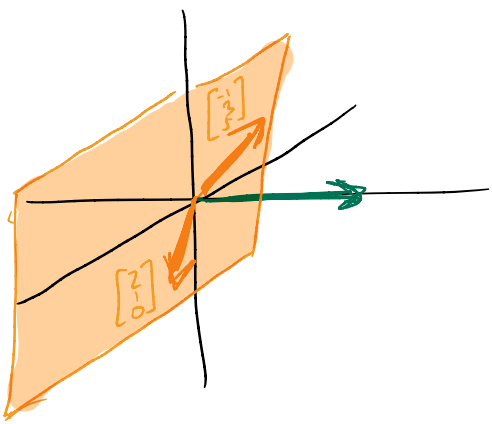
$$= \left\{ A \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \;\middle|\; \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{R}^n \right\}$$

$$= \{ Ax \mid x \in \mathbb{R}^n \}$$

$$= \{ b \in \mathbb{R}^m \mid Ax = b \text{ has a solution} \}$$

So $Ax=b$ has a solution if and only if $b \in C(A)$

Ex: $\begin{bmatrix} 2 & -1 \\ 1 & 3 \\ 0 & 5 \end{bmatrix} x = b$ has a solution if and only if $b \in \mathbb{R}^3$ lies in the plane $C(A)$



The vector $b = \begin{bmatrix} 0 \\ 5 \\ 0 \end{bmatrix}$ doesn't lie in the plane above ($b \notin C(A)$). Thus $\begin{bmatrix} 2 & -1 \\ 1 & 3 \\ 0 & 5 \end{bmatrix} x = \begin{bmatrix} 0 \\ 5 \\ 0 \end{bmatrix}$ has no soln.

Nullspace

Let A be an $m \times n$ matrix. The nullspace of A, $N(A)$, is the set of solutions to $Ax = 0$.

Key Feature: $N(A)$ is a subspace of $\mathbb{R}^n$ (where A is $m \times n$)

Proof:

- $A \cdot 0 = 0$ so $0 \in N(A)$
- Let $x, y \in N(A)$. Then $A(x+y) = Ax + Ay = 0$
  $\Rightarrow x+y \in N(A)$.
- Let $x \in N(A)$ and $c \in \mathbb{R}$.
  Then $A(cx) = c(Ax) = c0 = 0 \Rightarrow cx \in N(A)$ ∎

We will see that $N(A)$ is the span of some vectors.
To determine $N(A)$, we must solve $Ax = 0$.
Clearly 0 is always a solution.
In the case $m = n$:

- If A is invertible, then 0 is the only solution.
- If A is not invertible, then there must be infinitely many solutions.

Let's figure out how to describe the solution set as the span of some vectors.

Ex: Describe the nullspace of $A = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$.

Solve $Ax = 0$:

$$\begin{aligned} x_1 + 2x_2 &= 0 \\ 3x_1 + 6x_2 &= 0 \end{aligned} \longrightarrow \begin{aligned} x_1 + 2x_2 &= 0 \\ 0 &= 0 \end{aligned}$$ (infinite solns)

Every vector $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ satisfying $x_1 + 2x_2 = 0$ is a solution.

To describe this solution set as a span, rewrite $x_1 + 2x_2 = 0$ as $x_1 = -2x_2$. Then we have the general solution $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -2x_2 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} -2 \\ 1 \end{bmatrix}$ $x_2 \in \mathbb{R}$.

Special Solution (when $x_2 = 1$)

That is $N(A) = \text{Span}\left\{ \begin{bmatrix} -2 \\ 1 \end{bmatrix} \right\}$

In general, $N(A)$ = span of the "special solutions"

Let's make this more precise:

The Algorithm:

1. Reduce A to RREF (note that we don't need to reduce $[A|0]$ since last column remains 0)
   the variables corresponding to pivot columns are called pivot variables.
   the variables corresponding to non pivot columns are called free variables.

2. Rewrite the system in terms of the free variables and write down the general solution $x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ in terms of the special solutions $s_1, \dots, s_k$

3. Then $N(A) = \text{Span}\{s_1, \dots, s_k\}$.

Ex: $A = \begin{bmatrix} 1 & 1 & 2 & 4 \\ 1 & 2 & 2 & 5 \\ 1 & 3 & 2 & 6 \end{bmatrix}$. Describe $N(A)$.

First reduce to RREF: $\begin{bmatrix} 1 & 0 & 2 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ (pivot columns)

pivot variables: x_1, x_2
free variables: x_2, x_3

We have system: $x_1 + 2x_3 + 3x_4 = 0$
$x_2 + x_4 = 0$

Rewrite in terms of free variables: $x_1 = -2x_3 - 3x_4$
$x_2 = -x_4$

General solution:

special soln when $x_3 = 1, x_4 = 0$; special soln when $x_3 = 0, x_4 = 1$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -2x_3 - 3x_4 \\ -x_4 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -3 \\ -1 \\ 0 \\ 1 \end{bmatrix} \quad \text{where } x_3, x_4 \in \mathbb{R}$$

Hence $N(A) = \text{Span}\left\{ \begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ -1 \\ 0 \\ 1 \end{bmatrix} \right\}$ (plane in $\mathbb{R}^4$)